MULTISTAGE PRODUCTION FOR STOCHASTIC SEASONAL DEMAND*

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ABSTRACT

We consider the problem of production planning for a seasonal good which is produced in a multistage manner (e.g., when one or more components must be produced or purchased with a lead time that is long compared to the sales season). During the selling season, lost sales occur if demand cannot be satisfied; at the end of the season, leftover inventory incurs the usual overage cost. As the season progresses, the forecast of total demand is revised in light of current sales. The problem is to determine production quantities of the various components and assemblies at each period to minimize expected costs of underage and overage. If delivery is not required until the end of the selling (or "order-taking") season, then a dynamic programming formulation can produce the optimal decision rule. However, for the case in which delivery is required during the season, the associated dynamic programming formulation is computationally infeasible. The paper explores four heuristics for the latter problem and compares their cost performance in a numerical example. The most sophisticated heuristic produces expected profits which range from 3.2% to 5.5% of an upper bound on expected profit.

Introduction

We consider scheduling multistage production of seasonal goods with long lead components. Moreover, there is a series of production periods during which production decisions can be made. We assume that demand forecast revisions will be available in at least some of the production periods, thus affording the manufacturer an opportunity to improve his production strategy. Thus the manufacturer faces a complex stochastic sequential decision problem.

Some assembled goods have seasonal sales periods which may be short relative to the length of time required to produce or procure some of the assembly components. If demand is stochastic, then the quantity of the long lead components obtained can represent a major fixed commitment to the production of the finished good. Such a commitment must frequently be made well before sales results show the accuracy of the demand forecasts. If actual demand is higher than expected, then the quantity of the finished good to be sold is limited by the quantity of long lead components procured, and the shortage results in an opportunity cost through lost profits. On the other hand, if anticipated demand does not develop, there would be an oversupply of components whose cost could not be fully recovered.

Previous Research

Several aspects of the multistage production problem described above have been examined in three major related areas -- multistage production models, sequential production under uncertainty with forecast revisions, and the "newsboy" problem.

Algorithms for production scheduling have been presented for the finite horizon case, with decisions made at discrete points in time, and the infinite horizon case with constant demand. For the first case, discrete dynamic programming models have been developed by Zangwill [19], Veinott [18], and Love [10], which assume known but possibly time-varying demand and concave production and holding costs. Models for the infinite horizon case include [4], [5], [15] and [16]. For the seasonal problem considered in this paper, uncertain demand is a major component that must be explicitly treated. An attempt to decompose the scheduling decision into a deterministic portion with separate buffer stocks for forecast uncertainty would seem to discard the major element of the seasonal problem.

Previous work in sequential production under uncertainty with forecast revisions [3, 7, 8, 11] all deal with a seasonal good produced or procured in a single operation; the issue of multistage production or long lead components is absent. Although the basic "newsboy" problem (see [14]) is present

in our problem, the other aspects just mentioned make the direct newsboy solution inapplicable.

The Multistage Production System

In the multistage production system of this paper each stage or facility requires inputs from one or more immediate predecessors and in turn supplies one (and only one) immediate successor. We term the system an "assembly" structure if some stages have more than one predecessor as in Figure 1, or a "serial" structure if not.

We shall label a stage n, where n is an index from 1 to N, N being the final stage. Let a(n) be the index of the immediate successor of n, A(n) the set of indices of all successors, b(n) the set of indices of immediate predecessors and B(n) for all predecessors. In Figure 1, for example, a(1) = [3], b(3) = [1,2].

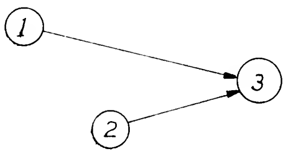

Figure 1

Associated with each stage of production will be the following cost parameters and decision variables.

c_n = the direct unit variable cost of processing at stage n

$C_n = c_n + \sum_{m \epsilon B(n)} c_m$, the cumulative direct unit variable cost of processing up through stage n

v_n = the salvage value added to a unit by processing at stage n

$V_n = v_n + \sum_{m \epsilon B(n)} v_m$ = the (cumulative) salvage value of a unit after stage n

z_n = the production or delivery lead time for stage n

$Z_n = z_n + \sum_{m \epsilon A(n)} z_m$ = the cumulative production lead time from the beginning of stage n to the end of the final stage N

$q_{n,t}$ = the <u>desired</u> production quantity at time period t for stage n (note that these units are not produced through stage n until a production lead time z_n has passed; thus, this variable represents a quantity of stage n material to be available at time period $t + z_n$)

$p_{n,t}$ = the <u>actual</u> amount of production actually started at stage n at time period t; $p_{n,t} \leq q_{n,t}$

$y_{n,t}$ = on-hand inventory of stage n material at time period t

$y_{n,t} = y_{n,t-1} + p_{n,t-z_n} - p_{a(n),t}$ for n = 1,...,N-1

D_t = demand for the assembled good in period t

$y_{N,t} = y_{N,t-1} + p_{N,t-z_n} - D_t$

Pr = the selling price of a unit of final product.

A Data-Generating Process for Seasonal Demand

We assume that the manufacturer knows the start and end points of the sales season. The sales season is divided into sales periods $t = 1,2,\ldots,T$. Further assume that he knows the seasonal pattern[1] of demands. Such a pattern might be expressed by defining for each period, t, an average fraction, s_t, of total seasonal demands that occur in this period, or as an average cumulative fraction, S_t, of total seasonal demands that occur through that period. Thus,

$$\sum_{i=1}^{t} s_i = S_t \; ; \qquad\qquad S_T = 1.00 \quad .$$

Once the sales pattern has been determined, the major issue for the manufacturer is to estimate the total level of demand over the season, D, for the good. It is assumed that the total level of demand for the season has a normal prior probability distribution with mean $\bar{D}(0)$ and variance $\sigma_D^2(0)$ when viewed prior to the sales season. That is, if the manufacturer forecasts total demand as a single point estimate

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[1] A manufacturer may gain knowledge of the typical sales pattern from empirical observation of sales of similar goods in previous seasons. References [3], [9], and [11] make this assumption.

$\bar{D}(0)$, the underlying level of total demand encountered will be Gaussian about a mean of $\bar{D}(0)$ with standard deviation $\sigma_D(0)$. From this the manufacturer can forecast demand for each period as

$$(1) \qquad \bar{D}_t(0) \;=\; s_t\bar{D}(0)$$

where $\bar{D}_t(0)$ is the forecasted demand for period t, the forecast being made at time period zero.

Following the model of Chang and Fyffe [3], assume that there is some "noise" in the demand pattern which causes deviations from the true seasonal pattern. Then the actual demand, D_t, for a period t is as follows:

$$(2) \qquad D_t = s_tD + e_t$$

where D = a Gaussian variable for the underlying total seasonal demand;

e_t = a random "noise" term, $N(0,\sigma_{pt})$, independent of D, and $E(e_t \cdot e_i) = 0$ for $i = 1,2,\ldots,T$ and $i \neq t$.

From this it follows that if

$$(3) \qquad e_p = \sum_{t=1}^{T} e_t$$

then

$$(4) \qquad E(e_p) = 0$$

and if σ_p^2 is the variance of the variable e_p, then

$$(5) \qquad \sigma_p^2 = \sum_{t=1}^{T} \sigma_{pt}^2 \,.$$

The above model of demand is the basic model to be studied. However, in order to reduce the number of system parameters, each σ_{pt} will be set by the following ad noc prespecified[2] relationship:

$$(6) \qquad \sigma_{pt} = \sqrt{s_t} \cdot \sigma_p \quad .$$

Note that given equation (6),

$$\sum_{t=1}^{T} \sigma_{pt}^2 = \sum_{t=1}^{T} s_t \cdot \sigma_p^2 = \sigma_p^2 \quad .$$

Thus only three parameters, $\overline{D}(0)$, $\sigma_D^2(0)$, and σ_p^2, plus the seasonal sales pattern $\{s_t\}$, are necessary to describe the demand data-generating process.

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[2]This scheme for "allocating" σ_p to the individual periods assigns a relatively lower σ_{pt} to periods of high demand than to periods with low demand. For example, define the ratio $\overline{D}(0)/\sigma_{pt}$, to be a measure of signal to noise for the period t, where $\overline{D}(0)$ is a measure of signal and σ_{pt} is the measure of noise. Then, by equation (6),

$$\overline{D}_t(0)/\sigma_{pt} = s_t\overline{D}(0)/\sqrt{s_t}\sigma_p = \sqrt{s_t} \cdot \overline{D}(0)/\sigma_p \ .$$

Thus the signal to noise ratio is higher for periods of high demand than for periods of low demand. "Noise" has a lesser effect on periods of robust demand than on periods of low demand.

Bayesian Forecast Revisions

After a producer makes an initial prior forecast for sales of a seasonal good, he will then make forecast revisions in light of his actual sales experience. Suppose that sales are generated by the demand model described above. As the season progresses, the actual demand observed in each period provides sample information about the true underlying level of total demand, D. Note that the decision maker has no control over the level of noise in the demand model. However, forecast revisions based on sample information will give improved estimates of the total level of demand underlying the seasonal sales pattern; that is, the standard deviation of a posterior distribution on D will tend to decline as forecast revisions are made.

The following variables will be used in developing a forecast revision scheme:

$\bar{D}(0)$ = the mean of the prior distribution of $\tilde{D}$

$\sigma_D^2(0)$ = the variance of the prior distribution of $\tilde{D}$

$\bar{D}(\tau)$ = an updated (posterior) mean of the distribution of $\tilde{D}$ as of the end of period τ

$\sigma_D^2(\tau)$ = the variance of the updated (posterior) distribution of $\tilde{D}$, as of the end of period τ

$\bar{D}_t(\tau)$ = an updated (posterior) mean of the distribution of the demand in period t, as of the end of period τ

σ_p^2 = the variance of the distribution of the total noise for the sales season; σ_p^2 is presumed known and will not change during the sales season

x_t = the actual cumulative demand up to and including period t.

A forecast revision on $\overline{D}(\tau)$ and $\sigma_D^2(\tau)$ is made by applying Bayes' theorem to determine a posterior probability distribution for D. We assume initial information about $\tilde{D}$ can be assessed through a Normal prior distribution. Given the prior parameters $\overline{D}(0)$ and $\sigma_D^2(0)$ and the system constants $\{s_t\}$ and σ_p^2, after observing the sample information that actual cumulative demand through period t is x_t, the mean and variance of the posterior (updated) distribution of total demand can be shown[3] to be:

$$(7) \qquad \overline{D}(t) = \frac{\overline{D}(0)\sigma_p^2 + (x_t/S_t)\cdot s_t\sigma_D^2(0)}{\sigma_p^2 + s_t\sigma_D^2(0)}$$

$$(8) \qquad \sigma_D^2(t) = \frac{\sigma_p^2\ \sigma_D^2(0)}{\sigma_p^2 + s_t\sigma_D^2(0)} \ .$$

The posterior mean in equation (7) is a weighted average of the prior mean and the pure demand extrapolation (x_t/s_t). Equation (8) shows that the posterior variance does not depend on the actual demand experience.

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[3]See Appendix.

Consider the case of a highly diffuse prior distribution on D, i.e., $\sigma_D(0)$ much greater than σ_p; then the posterior mean, equation (7), of the distribution of total demand becomes

(9) $\quad \bar{D}(t) = x_t/S_t$

which is pure extrapolation of actual demand to date.

In general, the forecast of demand for each period is updated as

(10) $\quad \bar{D}_t(\tau) = s_t\bar{D}(\tau)$

(11) $\quad \sigma_t^2(\tau) = s_t^2\ \sigma_D^2(\tau) + s_t\sigma_p^2$

where $\bar{D}_t(\tau)$ is the forecast of expected demands in period t as estimated as the end of period τ, and $\sigma_t^2(\tau)$ is the variance of the distribution of demand in period t as estimated at the end of period τ. $\sigma_t^2(\tau)$ includes uncertainty both from the random noise in the sales pattern and the uncertainty from the total underlying level of demand.

Combining the Forecast and Decision Models

Now consider that the production network described earlier has been combined with the Bayesian demand and forecast revision data-generating process just described. Then our problem is a sequential decision problem, as follows. At the end of time period τ, the information available to us includes updated demand parameters $\bar{D}(\tau)$ and $\sigma_D^2(\tau)$ (see equations (7) and (8)), and al previous production decisions $p_{n,t}$ for $t=1,2,\ldots,\tau$ (recall $p_{n,t}$ is the

variable representing the <u>actual</u> quantity of units ordered into stage n of the manufacturing process at period t).

Could this problem be formulated as a dynamic programming problem with a sufficiently small state space dimensionality so that computational results could be obtained? This depends on the assumptions one is willing to make. In particular, if the "demands" which occur are actually orders which do not require delivery until the end of the order-taking season[4], and if production capacity is assumed unlimited, then one would work backwards and find, for each stage of production, the <u>last</u> time period in which production could take place and still meet the end-of-season deadline. These "decision points", one for each stage, would represent the only times when, under the stated assumptions, one would have to commit himself to some production. Moreover, a sufficient state variable for inventory would be the current cumulative amount of production possibly attainable (due to already-made commitments on raw materials). Thus the state space would contain the latest upper limit on production and the updated mean and variance of the posterior distribution on $\tilde{D}$. With a three-dimensional state space and a modest number of decision points, such a dynamic programming formulation could be solved computationally, although both the programming and the computation time would be nontrivial. However, if one wishes to relax the assumption

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[4]This assumption has been called the "terminal-delivery" assumption; see [8].

of no delivery required during the season, then it becomes necessary to record the specific past decisions, $p_{n,t}$, and the state space becomes much too large for any computational results. We make the latter assumption and proceed to explore various heuristics on a sample problem.

The Heuristic Decision Rules

The decision variables for the problem are $q_{n,t}$, $n = 1,2,\ldots,N$, $t = 1,2,\ldots,T$, the desired level of production to initiate at each stage of the process at each time period. In any given period t, the amount actually started in production, $p_{n,t}$, may be smaller than the desired level of product if "excess inventory" for the stage exists (on hand or in process). By excess inventory, we refer to previously initiated production which, after subsequent forecast revisions, is now in excess of "updated" desired quantities. Specifically, let:

$q_{n,\tau}^{(t)}$ = updated (as of time t) desired production quantity to have been ordered into production at time τ; $t > \tau$.

$E_{n,t}$ = excess inventory (above newly updated desired quantities) on hand and on order at stage n at time t.

Excess inventory, $E_{n,t}$, is calculated recursively as follows:

$$E_{n,t-z_n} \equiv \text{Max} \ \{0, \ y_{n,t} - q^{(t)}_{n,t-z_n}\}$$

$$E_{n,\tau} \ = \ \text{Max} \ \{0, \ E_{n,\tau-1} + p_{n,\tau} - q^{(t)}_{n,\tau}\}$$

$$\text{for } \tau = t-z_n+1 \text{ to } t.$$

The complexity of the recursion results from the fact that excesses are carried forward, but deficiencies cannot be; we assume sales are lost in any period with insufficient inventory. Thus we net out from desired production any excess: $p_{n,t} = q_{n,t} - E_{n,t}$.

However, if sufficient input materials for the stage do not exist, the production quantity must also be reduced. Therefore we may set

$$(12) \qquad p_{n,t} \ = \ \min \ \{(q_{n,t} - E_{n,t}), \ \min_{m\varepsilon b(n)} (y_{m,t-1} + p_{m,t-z_m})\}.$$

Equation (12) indicates that the production quantity should not exceed the net desired quantity ($q_{n,t} - E_{n,t}$) after excess inventory is considered; and it cannot exceed the inventory (on hand plus just-produced) of the most limiting of the immediate predecessors {b(n)} of stage n.

Finally, one further restriction will be placed on production quantities $p_{n,t}$. If $Z_m > Z_n$, then $p_{n,t}$ should be

restricted so as not to produce more material than can be subsequently assembled with stage m inventory (on hand plus on order) when the two components meet.

We now present four heuristic rules for the determination of $q_{n,t}$ and assume the relations just described are used to determine $p_{n,t}$.

Heuristic H1 simply sets desired production quantities of stage n equal to expected demand in period $t+Z_n$ (when the corresponding assembled good is available):

$$(13) \quad q_{n,t} = s_{t+Z_n} \hat{D}(t-1)$$

where $\hat{D}(t-1)$ is the current (through time period t-1) forecast revision for the total underlying level of demand. Thus H1 ignores the "newsboy" aspect of the problem. We include this heuristic primarily to give a basis of comparison for the following, more sophisticated heuristics.

The second heuristic, H2, is a direct application of the newsboy solution to the individual periods of the sales season. For the single period case, with a normal distribution, the optimal stock quantity, q, is

$$q = \mu + \delta \cdot \sigma$$

where μ = process average

σ = the standard deviation of the distribution

$\delta = F^{-1}[C_u/C_u + C_o]$

F^{-1} = inverse of the cumulative standardized normal distribution

C_u = unit cost of underage

C_o = unit cost of overage.

In our application

$$C_u = Pr - C_N$$

$$C_o = C_N - V_N$$

$$\mu = s_{t+Z_n} \hat{D}(t-1) \qquad \text{[see equation (10)]}$$

and $$\sigma = \sigma_{n,t} = \sqrt{\sigma^2_{t+Z_n}(t-1)} \qquad \text{[see equation (11)]}$$

Now $q_{n,t}$ is determined in H2 as:

(14) $$q_{n,t} = s_{t+Z_n} \hat{D}(t-1) + \delta\sigma_{n,t}.$$

Heuristic H2 always uses the cost of overage for the assembled good, while in fact any overage early in the season or early in the assembly process would generally involve a lesser amount.

Heuristic H3 considers a cost of overage for each individual stage, not for just the finished good. The cost of overage includes the incremental cost of overage for the stage (n) plus a heuristically weighted portion of the incremental cost of overage for all stages {m} in the network for which $Z_n \geq Z_m$. The weights are arbitrarily selected to be a function of the relative uncertainty faced by the various stages at the last time period for which they can begin production to meet demand in period $t+Z_n$. The cost of overage is defined as:

(15) $$C_{o_{n(t)}} = \sum_{m=1}^{N} [f_n(m) \quad (C_m - V_m)]$$

(16) $$\text{where } f_n(m) = \begin{cases} [\sigma_{m,(t+Z_n-Z_m)}/\sigma_{n,t}] & \text{if } Z_n \geq Z_m \\ 0 & \text{otherwise} \end{cases}.$$

Then,

(17) $$\delta_{n,t} = F^{-1}[C_u/(C_u + C_{o_{n(t)}})]$$

is used for equation (14) in place of δ to determine $q_{n,t}$.

Note that no weight is given to stages for which production decisions are fixed for period $t+Z_n$; i.e., for stages m for which $Z_m > Z_n$. Heuristic H3 has one final deficiency; whatever percentile of demand is "protected against" through $[C_u/(C_u + C_{o_{n,t}}]$, this percentile is double-counted, triple-counted, etc., since demand in fact will randomly fluctuate about its average.

Heuristic H4 is similar to H3, but avoids the double- and triple-counting, etc., by considering total demand over periods t to $t+Z_n$ rather than demand for just the period $t+Z_n$. It exploits the fact that overproduction in early periods may be consumed in later periods and thus the risk of overage cost is low. This is achieved by setting

$$(18) \qquad q_{n,t} = Q_{n,t}(t+Z_n) - Q_{n,t}(t+Z_n-1)$$

where $Q_{n,t}(\tau)$ = the cumulative desired production quantity for stage n for time periods t through τ.

$$(19) \qquad Q_{n,t}(\tau) = \sum_{i=t}^{t+Z_n} s_i\hat{D}(\tau-1) + \delta^*_{n,t}\sigma C_{n,t}(\tau)$$

where $\sigma C_{n,t}(\tau)$ is the standard deviation of demand for the cumulative[4] period from t to τ and $\delta^*_{n,t}$ will be defined shortly.

The cost of overage is now defined as

$$(20) \qquad C^*_{o,n(t)} = \left[\sum_{m=1}^{N} f^*_n(m) \times (C_m - V_m)\right] \left[\frac{S_{t+Z_n} - S_{t-1}}{S_T - S_{t-1}}\right]$$

$$\text{and } f^*_n(m) = \begin{cases} \dfrac{\sigma C_{m,T-Z_m}}{\sigma C_{n,T-Z_n}} & \text{if } Z_n \geq Z_m \\ 0 \text{ otherwise} & . \end{cases}$$

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[4]The quantity $\sigma C_{n,t}(\tau)$ may be derived from equation (11).

The last term in equation (20) arbitrarily decreases the overage cost in periods prior to $T-Z_n$, to reflect the fact that excess production can be absorbed in later periods. Then

(21) $\delta^*_{n,t} = F^{-1}[C_u/C_u + C^*_{o,n(t)}]$

and finally, substitution of (21) into (19) allows (18) to produce a value for $q_{n,t}$.

Simulation Tests of the Heuristics

To test the heuristics developed in the previous section a simulation model was developed. The 3-stage network of Figure 1 was used with the seasonal demand of Table 1. The basic cost and demand parameters are given in Table 2. All profit data reported are the average of 50 trials.

Period t	1	2	3	4	5	6	7	8	9	10	11	12
s_t	0	0	0	0	0	0	.05	.10	.20	.25	.25	.15

Table 1: Seasonal Demand Pattern

Stage n =	1	2	3		
C_n	4	4	12	Price Pr =	\$16.00
V_n	3	3	9	$\hat{D}(0)$ =	20,000
z_n	5	1	0	$\sigma_D(0)$ =	6,000
				σ_p =	2,000

Table 2: Basic Problem Parameters

Table 3 shows the results of three simulation tests of 50 trials each. H4 provides the best results in all tests.

Run	Heuristic	Average Profit	% Above H1
1	1	62,337	0.0
	2	64,335	3.2
	3	67,042	7.5
	4	68,862	10.5
2	1	59,523	0.0
	2	61,410	3.2
	3	63,862	7.3
	4	66,167	11.1
3	1	60,413	0.0
	2	62,406	3.3
	3	64,877	7.4
	4	66,579	10.2

Table 3: Results of Three Simulation Tests

(σ_D = 6000, σ_p = 2000 throughout)

Table 4 shows the test results for varying levels of market uncertainty, i.e., with varying values for σ_D and σ_p. (Table 3 contained σ_D = 6000 and σ_p = 2000 throughout.) As expected, the average profit under all heuristics declines as market uncertainty increases. But H4 improves relative to H1 from only 4.6 percent better than H1 for low levels of uncertainty to 16.9 percent better for higher levels of uncertainty. This is strong evidence that H4 is effective in handling market uncertainty.

Sigma	H	Profit	% Above H1
σ_D=3000	1	70,647	0.0
	2	71,664	1.4
σ_p=1000	3	72,930	3.2
	4	73,950	4.6
σ_D=9000	1	50,735	0.0
	2	53,283	5.0
σ_p=3000	3	56,529	11.4
	4	59,330	16.9

Table 4: Effect of Varying Market Uncertainty

H4 clearly proved to be the most effective of the heuristics tested. It recognizes the fact that the economic risk of overproduction in early periods of the sales season is not as great as in the later periods. H4 also is responsive to market uncertainty, economic risk, relative stage costs, and relative lead times. As do all the heuristics tested, H4 makes use of the forecast revisions and actual demand experience, to adjust production quantities.

In this section, we have discussed the relative performance of four heuristics. We now turn to a discussion of upper bounds on expected profit so that absolute measures of performance can be made.

Upper Bounds on Expected Profit

A simple upper bound on expected profit can be determined as profit, per unit, $(Pr - C_N)$, times expected demand, $\bar{D}$. This profit could be achieved only with perfect forecasts. For our sample problem that would be \$80,000. However, uncertainty in demand (or equivalently, forecast error) will reduce expected profit from this level. For the single stage, single period newsboy problem, with a demand distribution $N(\bar{D},\sigma)$,

the expected profit for an optimal policy is[5]

(22) $\overline{P} = (Pr - C_N)\overline{D} - \sigma(C_u + C_o)f_N(Q)$

where $f_N(Q)$ = the value of the normal probability density function for demand at the optimal order quantity Q.

This relationship can be applied to find an upper bound on expected profit for the multistage, multiperiod problem. We will apply the newsboy bound to each stage, n, of the multistage problem, considering only those costs prior to and including the production of stage n, and using as a measure of uncertainty in demand that uncertainty remaining in the last period at which stage n can begin production, $T-Z_n$.

We now compute the upper bound, B_n, as

(23) $B_n = (Pr - C_N)\overline{D} - L_n$

where

(24) $L_n = (\sigma C_{n,T-Z_n}) \quad (C_u + \sum_{j \varepsilon G(n)} c_j - v_j) \quad f(q_{n,T-Z_n})$

and G(n) is the set of all stages {m} such that $Z_m \geq Z_n$. Then $B^* = \min_n \{B_n\}$ is an overall bound on expected profit, but a bound which does not include losses that may arise from underage or overage in later production stages, nor does it include potential losses from demand prior to the production of

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[5] See [14], p. 302.

the final batch for the bounding stage.

The bound was applied to the sample problem reported in Table 3 for which H4 gave expected profits of \$68,862, \$66,167, and \$66,579. The computed bounds were B1 = 71,150, B2 = 76,560, and B3 = 77,720; thus B1 is the tightest upper bound. It is noteworthy that the best result for H4 was only 3.2% below the tightest bound and the worst was only 5.5% below; and the bound calculated does not include all sources of possible loss.

Appendix

Derivation of Posterior Distribution for Forecast Revision

Given prior parameters $\bar{D}(0)$ and σ_D^2 for a demand forecast, it is necessary to determine posterior parameters $\bar{D}(t)$ and $\sigma_D^2(t)$ for the revised forecast at the end of period t. The demand model is:

$$D_t = s_t D + e_t$$

where

D_t = the demand in period t.

D = the underlying total level of demand. D has a Gaussian distribution with mean $\bar{D}(0)$ and variance $\sigma_D^2(0)$.

e_t = zero mean Gaussian noise in the demand for period t with standard deviation $\sqrt{s_t} \cdot \sigma_p$.

After t periods of the sales season, based on sample information of x_t cumulative demands, Bayes' theorem can be used to obtain a posterior mean, $\bar{D}(t)$ and variance $\sigma_D^2(t)$, for the beginning of the next sales period. Bayes' theorem is, in our notation,

$$\text{Prob}(D/x_t) = \frac{\text{Prob}(D) \cdot \text{Prob}(x_t/D)}{\int_{-\infty}^{\infty} \text{Prob}(D) \cdot \text{Prob}(x_t/D)\, dD} .$$

The sample information is generated by:

$$x_t = S_t D + \sum_{i=1}^{t} e_i . \tag{25}$$

Thus, the conditional distribution of x_t given D is Normal with mean $S_t D$ and variance $S_t \sigma_p^2$, and we may write

(26) $$\mathrm{Prob}(x_t/D) = (2\pi S_t)^{-\frac{1}{2}} \cdot \frac{1}{\sigma_p} \exp(-\tfrac{1}{2}(x_t - S_t D)^2 / S_t \sigma_p^2) \quad ;$$

By definition

(27) $$\mathrm{Prob}(D) = (2\pi)^{-\frac{1}{2}} \cdot \frac{1}{\sigma_D(0)} \exp(-\tfrac{1}{2}(D - \bar{D}(0))^2 / \sigma_D^2(0) \quad .$$

Substituting these expressions into Bayes' theorem, the denominator and the constants in the numerator reduce to a simple normalizing factor, leaving

(28) $$\mathrm{Prob}(D/x_t) = K \cdot \exp[-\tfrac{1}{2}(D - \bar{D}(0))^2 / \sigma_D^2(0) - \tfrac{1}{2}(x_t - S_t D)^2 / S_t \sigma_p^2]$$

where K is the normalizing constant. Letting E be the exponent of the above equation, and combining terms and arranging according to powers of D:

(29) $$E = (-\tfrac{1}{2})(1/\sigma_p^2 \sigma_D^2(0)) \; [D^2(\sigma_p^2 + S_t \sigma_D^2(0)) - 2D(\sigma_p^2 \bar{D}(0) + x_t \sigma_D^2(0)) + (\text{other terms not including } D)].$$

The above form of E is precisely the functional form of the Gaussian exponent with general form

$$E_{\text{Gaussian}} = -\tfrac{1}{2}(1/\sigma^2)(X^2 - 2X\mu + \mu^2) = -\tfrac{1}{2}(1/\sigma^2)(X-\mu)^2.$$

Thus the posterior parameters can be extracted from the exponent of the posterior distribution as follows:

(22) $$\bar{D}(t) = (\sigma_p^2 \bar{D}(0) + \sigma_D^2(0) x_t)/(\sigma_p^2 + S_t \sigma_D^2(0))$$

(23) $$\sigma_D^2(t) = \sigma_p^2 \sigma_D^2(0)/(\sigma_p^2 + S_t \sigma_D^2(0)).$$

These equations are consistent with the corresponding ones of reference [3]; they differ slightly because of the assumption made in the present paper's equation (6).

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